Announcements

1) Quiz Thursday over 13.1, 13.2 (limits, derivatives, and tangent lines)

2) Exam 1 next week

Vector Functions

(Chapter 13)

A vector function is a function $f: \mathbb{R} \to \mathbb{R}^n$ such that the images $f(t)$ are vectors in $\mathbb{R}^n$.

Example 1: Some vector functions are

$$f(t) = \langle 2+3t,\ 9-5t,\ 10t \rangle$$

(graph is a line)

$$g(t) = \langle \cos(t),\ \sin(t),\ t \rangle$$

(graph = ?)

Vector Functions and Space Curves

(Section 13.1)

Graphs and such

Just like lines, the graph of a vector-valued function is made up of all terminal points of vectors in the range of the function.

Example 2:

Graph

$$f(t) = \langle \cos(t), \sin(t), t \rangle$$

Mathematica

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ParametricPlot3D[{Cos[t], Sin[t], t},
                 {t, t0, t1}]
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where t_0 and t_1 are numbers you choose.

This is the graph of a helix

Limits

If $f(t) = \langle x(t), y(t), z(t) \rangle$

is a vector function, we

let

$$\boxed{\lim_{t \to t_0} f(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle}$$

provided all these limits exist!

Note:

If even one of the limits at $t \to t_0$ of $x(t)$, $y(t)$, and $z(t)$ does not exist, then $\lim_{t \to t_0} f(t)$ does not exist!

Example 3:

Find

$$\lim_{t \to 1} \left\langle \frac{t^3 - t^2 + t - 1}{t-1}, \frac{\sin(5t-5)}{7t-7}, t^{\frac{1}{t-1}} \right\rangle$$

or show the limit does not exist.

$$x(t) = \frac{t^3 - t^2 + t - 1}{t-1}$$

Plugging in $t=0$ gives

$\frac{0}{0}$, so the quotient

is indeterminate.

We can factor

$$t^3-t^2+t-1=(t-1)(t^2+1),$$

so

$$\lim_{t\to 1}\frac{t^3-t^2+t-1}{t-1}=\lim_{t\to 1}\frac{\cancel{(t-1)}(t^2+1)}{\cancel{t-1}}$$

$$=\lim_{t\to 1}(t^2+1)=\boxed{2}$$

$\lim_{t \to 1} x(t)$ exists, so we turn

to $\lim_{t \to 1} \dfrac{\sin(5t-5)}{7t-7}$.

Plugging in $t=1$ gives

$$\frac{\sin(0)}{0} = \frac{0}{0}$$, again

indeterminate, but we can't

do any factoring this time!

We use l'Hopital's rule:

If $\lim_{t \to a} f(t) = \lim_{t \to a} g(t) = 0$

(or both limits are $\pm\infty$),

then $$\boxed{\lim_{t \to a} \frac{f(t)}{g(t)} = \lim_{t \to a} \frac{f'(t)}{g'(t)}}$$

Then $$\lim_{t \to 1} \frac{\sin(5t-5)}{7t-7}$$

chain rule

$$= \lim_{t \to 1} \frac{\cos(5t-5) \cdot 5}{7}$$

$$= \boxed{5/7}$$

Now $\lim_{t \to 1} y(t)$ also exists,

so we only have to find

$\lim_{t \to 1} z(t)$ or show it

does not exist.

Plugging in $t=1$ to

$z(t) = t^{\frac{1}{t-1}}$ gives

$1^{\frac{1}{0}}$, again indeterminate.

This time we don't have a quotient to use l'Hopital's rule, so we have to make one!

$$t^{\frac{1}{t-1}} = e^{\ln\left(t^{\frac{1}{t-1}}\right)}$$

$$= e^{\frac{\ln(t)}{t-1}} \quad \text{(log rules)}$$

If we can find $L = \lim_{t \to 1} \frac{\ln(t)}{t-1}$,

the limit will be e^L.

Using l'Hopital's rule again,

$$\lim_{t \to 1} \frac{\ln(t)}{t-1} = \lim_{t \to 1} \frac{\frac{1}{t}}{1}$$

$$= 1, \text{ so}$$

$$\lim_{t \to 1} z(t) = e^1 = \boxed{e}$$

Finally,

$$\boxed{\lim_{t \to 1} f(t) = \langle 2, 5/7, e \rangle}$$

Continuity

We won't really do anything with this, but here's the definition: if f is a vector-valued function, we say that f is **continuous** at $t = t_0$ if

$$\boxed{\lim_{t \to t_0} f(t) = f(t_0)}$$

Derivatives and Integrals of Vector Functions

(Section 13.2)

We will first look at secant lines, then make the step to tangent lines.

Example 4: If

$f(t) = \langle \cos(t), \sin(t), t \rangle,$

find the equation of the line through the terminal points of $f(0) = \langle 1, 0, 0 \rangle$ and $f(\pi/2) = \langle 0, 1, \pi/2 \rangle$

The direction vector is given by subtracting $(1,0,0)$ from $(0,1,\pi/2)$.

We get the vector

$\langle -1, 1, \pi/2 \rangle$, so that

the equation for the line

is

$$\boxed{\langle 1, 0, 0 \rangle + t\langle -1, 1, \pi/2 \rangle}$$

Tangent Lines

If $f(t) = \langle x(t), y(t), z(t) \rangle$,

then we can construct the secant line through the terminal points of

$$f(t_0) = \langle \underbrace{x(t_0)}_{x_0}, \underbrace{y(t_0)}_{y_0}, \underbrace{z(t_0)}_{z_0} \rangle$$

and

$$f(t_1) = \langle \underbrace{x(t_1)}_{x_1}, \underbrace{y(t_1)}_{y_1}, \underbrace{z(t_1)}_{z_1} \rangle$$

provided $f(t_0) \neq f(t_1)$

The line will have the form

$$\langle x_0, y_0, z_0 \rangle + s \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + \frac{s}{t_1 - t_0} \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + s \left\langle \frac{x_1 - x_0}{t_1 - t_0}, \frac{y_1 - y_0}{t_1 - t_0}, \frac{z_1 - z_0}{t_1 - t_0} \right\rangle$$

Since $\frac{1}{t_1 - t_0} \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ is

a vector parallel to $\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

Now take the limit as $t_1 \to t_0$.

Since $\langle x_0, y_0, z_0 \rangle$ has constant coordinates, the limit passes by to

$$s\left\langle \lim_{t_1 \to t_0} \frac{x_1 - x_0}{t_1 - t_0}, \lim_{t_1 \to t_0} \frac{y_1 - y_0}{t_1 - t_0}, \lim_{t_1 \to t_0} \frac{z_1 - z_0}{t_1 - t_0} \right\rangle$$

$$= s\left\langle \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0}, \lim_{t_1 \to t_0} \frac{y(t_1) - y(t_0)}{t_1 - t_0}, \lim_{t_1 \to t_0} \frac{z(t_1) - z(t_0)}{t_1 - t_0} \right\rangle$$

$$= s\langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

provided all these limits exist!

Then the equation of the tangent line at $t = t_0$ is

$$\boxed{\langle x(t_0), y(t_0), z(t_0) \rangle + s \langle x'(t_0), y'(t_0), z'(t_0) \rangle}$$

Example 5: Find the tangent line to

$$k(t) = \langle \ln(3^{t^3}), e^{\log_4(t)}, t^t \rangle$$

at $t=1$

$x(t) = \ln(3^{t^3}) = t^3 \ln(3)$ by log properties, so

$x'(t) = 3t^2 \ln(3)$, and

$x(1) = \ln(3)$, $x'(1) = 3\ln(3)$

$$y(t) = e^{\log_4(t)}$$

$$= e^{\frac{\ln(t)}{\ln(4)}}$$

$$= e^{\ln\left(t^{\frac{1}{\ln(4)}}\right)}$$

$$= t^{\frac{1}{\ln(4)}}, \quad \text{so}$$

$$y'(t) = \frac{1}{\ln(4)}\, t^{\left(\frac{1}{\ln(4)}-1\right)}, \text{ and}$$

$$y(1) = 1, \; y'(1) = \frac{1}{\ln(4)}$$

The hardest derivative is last:

$$z(t) = t^t$$
$$= e^{\ln(t^t)}$$
$$= e^{t\ln(t)}$$

Then

$$z'(t) = e^{t\ln(t)} \underbrace{\frac{d}{dt}\left(t\ln(t)\right)}_{\text{Chain rule}}$$
$$= e^{t\ln(t)}\left(\ln(t)+1\right)$$
$$= t^t\left(\ln(t)+1\right), \text{ and}$$

$z(1) = 1, \quad z'(1) = 1.$

Putting all this together,

the tangent line is given by

$$\langle x(1), y(1), z(1) \rangle + s\langle x'(1), y'(1), z'(1) \rangle$$

$$= \boxed{\left\langle \ln(3), 1, 1 \right\rangle + s\left\langle 3\ln(3), \frac{1}{\ln(4)}, 1 \right\rangle}$$

The Derivative

If it exists, the derivative of a vector-valued function $f(t) = \langle x(t), y(t), z(t) \rangle$ is the function

$$\boxed{f'(t) = \langle x'(t), y'(t), z'(t) \rangle}$$

We can then restate the formula for the tangent line at $t = t_0$ as

$$\boxed{f(t_0) + s f'(t_0)}$$